Solving Acoustical Problems Using the Laplace-Transformation

report - in making

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1. Introduction

1.1. Basics

The basic equations describing the acoustical fields are of the form of Eq. (1), where Eq. (1a) is from the conservation of mass, Eq. (1b) is by Euler from the conservation of momentum and finally (1c) is from the assumption that pressure is a function of density and entropy [1 - Pierce]:

$$\frac{\partial \rho}{\partial t} + \overline{\nabla} \cdot \left(\rho \overline{v} \right) = 0 \qquad (1.1.1a)$$

$$\rho \left[\frac{\partial \overline{v}}{\partial t} + (\overline{v} \cdot \nabla) \overline{v} \right] = -\overline{\nabla} p \qquad (1.1.1b)$$

$$p = p(\rho, s) \qquad (1.1.1c)$$

Introducing the ambient-field variables, we can write Eqs. (1) using the sum of ambient and a varying pressure, density and velocity.

$$\frac{\partial(\rho_0 + \rho')}{\partial t} + \overline{\nabla} \cdot \left[(\rho_0 + \rho') \overline{v}' \right] = 0 \qquad (1.1.2a)$$

$$(\rho_0 + \rho') \left(\frac{\partial \overline{v}'}{\partial t} + (\overline{v}' \cdot \nabla) \overline{v}' \right) = -\overline{\nabla}(p_0 + p') \qquad (1.1.2b)$$

$$p_0 + p' = p(\rho_0 + \rho', s_0) \qquad (1.1.2c)$$

Here $\overline{v}_0 = 0$, i.e. the fluid is not moving. From Eqs. (2) we can get a linear approximation by neglecting the higher order and mixed terms from Eq. (2a) and (2b), and by using only the first term of the Taylor-series expansion from the approximation of Eq. (2c).

$$\frac{\partial \rho'}{\partial t} + \rho_0 \overline{\nabla} \cdot \overline{v} = 0 \qquad (1.1.3a)$$

$$\rho_0 \frac{\partial \overline{v}'}{\partial t} = -\overline{\nabla} p' \qquad (1.1.3b)$$

$$p' = \left(\frac{\partial p}{\partial \rho} \right)_0 = c^2 \rho' \qquad (1.1.3c)$$

Deleting the primes on pressure and particle velocity, and using Eq. (3c) the basic equation for our investigations are:

$$\frac{1}{c^2}\frac{\partial p}{\partial t}+\rho_0\overline{\nabla}\cdot\overline{v}=0 \tag{1.1.4a}$$

$$\rho_0\frac{\partial\overline{v}}{\partial t}=-\overline{\nabla}p \tag{1.1.4b}$$

Here the pressure and the particle velocity are functions of x, y and z in a Cartesian coordinate-system, and of time:

$$p=p(x,y,z,t)$$
$$\overline{v}=\overline{v}(x,y,z,t)=v_x(x,y,z,t)\cdot\overline{e}_x+v_y(x,y,z,t)\cdot\overline{e}_y+v_z(x,y,z,t)\cdot\overline{e}_z\cdot$$

Writing Eq. (4) in a detailed form:

$$\frac{1}{c^2}\frac{\partial p}{\partial t}+\rho_0\left[\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial x}+\frac{\partial v_z}{\partial x}\right]=0 \tag{1.1.5a}$$

$$\rho_0\left[\frac{\partial v_x}{\partial t}\overline{e}_x+\frac{\partial v_y}{\partial t}\overline{e}_y+\frac{\partial v_z}{\partial t}\overline{e}_z\right]=-\left[\frac{\partial p}{\partial x}\overline{e}_x+\frac{\partial p}{\partial y}\overline{e}_y+\frac{\partial p}{\partial z}\overline{e}_z\right] \tag{1.1.5b}$$

We can write Eq. (5b) separated for each vector-component to get with Eq. (5a) an equation system with 4 equations for the 4 unknown field variables.

$$\rho_0\frac{\partial v_x}{\partial t}=-\frac{\partial p}{\partial x} \tag{1.1.6a}$$

$$\rho_0\frac{\partial v_y}{\partial t}=-\frac{\partial p}{\partial y} \tag{1.1.6b}$$

$$\rho_0\frac{\partial v_x}{\partial t}=-\frac{\partial p}{\partial x} \tag{1.1.6c}$$

1.2. The Laplace-transformation

The definitions of the Laplace-transformation and of the inverse Laplace-transformation are:

$$F(s)=\int_{-0}^{\infty}f(x)e^{-sx}dx \quad \text{, and} \quad f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(s)e^{sx}ds \tag{1.2.1}$$

The Laplace-transformation is linear and has very similar properties to the well-known Fourier-transformation, though doesn’t presumes the transformed function to be stationary.
An important property of this transformation is for the derivative of a transformed function to be:

$$G(s)=\int_{-0}^{\infty}\frac{\partial f(x)}{\partial x}e^{-sx}dx=s\cdot F(s)-f(0) \tag{1.2.2}$$

Here $f(0)$ denotes the initial value of the function.

1.3. Using the Laplace-transformation

Using the Laplace-transformation described above, we can derive from Eq. (1.1.5a) and Eqs. (1.1.6) the following transformed equation system, where x, y, z and t are transformed to g, h, l and s respectively:

$$\frac{1}{c^2}\left[s\cdot P(s,g,h,l)-p(0,g,h,l)\right]+$$
$$+\rho_0\left[g\cdot V_x(s,g,h,l)-v_x(s,0,h,l)+h\cdot V_y(s,g,h,l)-v_y(s,g,0,l)+l\cdot V_z(s,g,h,l)-v_z(s,g,h,0)\right]=0 \tag{1.3.1a}$$

$$\rho_0\left[s\cdot V_x(s,g,h,l)-v_x(0,g,h,l)\right]=-g\cdot P(s,g,h,l)+p(s,0,h,l) \tag{1.3.2a}$$
$$\rho_0\left[s\cdot V_y(s,g,h,l)-v_y(0,g,h,l)\right]=-h\cdot P(s,g,h,l)+p(s,g,0,l) \tag{1.3.2b}$$
$$\rho_0\left[s\cdot V_z(s,g,h,l)-v_z(0,g,h,l)\right]=-l\cdot P(s,g,h,l)+p(s,g,h,0) \tag{1.3.2c}$$

Let us denote the initial values as follows:

$$p(0,g,h,l)=A_1 \tag{1.3.3a}$$
$$p(s,0,h,l)=A_2 \tag{1.3.3b}$$
$$p(s,g,0,l)=A_3 \tag{1.3.3c}$$
$$p(s,g,h,0)=A_4 \tag{1.3.3d}$$
$$v_x(0,g,h,l)=B_1 \tag{1.3.3e}$$
$$v_x(s,0,h,l)=B_2 \tag{1.3.3f}$$
$$v_y(0,g,h,l)=D_1 \tag{1.3.3g}$$
$$v_y(s,g,0,l)=D_2 \tag{1.3.3h}$$
$$v_z(0,g,h,l)=E_1 \tag{1.3.3i}$$
$$v_z(s,g,h,0)=E_2 \tag{1.3.3j}$$

Using Eqs. (3), Eq. (1) and Eqs. (2) may be written in a shorter form:

$$\frac{1}{c^2}\left[s\cdot P-A_1\right]+\rho_0\left[g\cdot V_x-B_2+h\cdot V_y-D_2+l\cdot V_z-E_2\right]=0 \tag{1.3.4}$$
$$\rho_0\left[s\cdot V_x-B_1\right]=-g\cdot P+A_2 \tag{1.3.5a}$$
$$\rho_0\left[s\cdot V_y-D_1\right]=-h\cdot P+A_3 \tag{1.3.5b}$$
$$\rho_0\left[s\cdot V_z-E_1\right]=-l\cdot P+A_4 \tag{1.3.5c}$$

Expressing the transformed velocities for each vector-component from Eqs. (5) gives:

$$V_x=\frac{1}{s}\left[\frac{-g\cdot P+A_2}{\rho_0}+B_1\right] \tag{1.3.6a}$$
$$V_y=\frac{1}{s}\left[\frac{-h\cdot P+A_3}{\rho_0}+D_1\right] \tag{1.3.6b}$$
$$V_z=\frac{1}{s}\left[\frac{-l\cdot P+A_4}{\rho_0}+E_1\right] \tag{1.3.6c}$$

With Eqs. (6) from Eq. (4) for the transformed pressure we get

$$P(s,g,h,l)=\frac{\left[\frac{1}{c^2}A_1+\rho_0\left(B_2+D_2+E_2\right)\right]\cdot s-\left(A_2+\rho_0 B_1\right)\cdot g-\left(A_3+\rho_0 D_1\right)\cdot h-\left(A_4+\rho_0 E_1\right)\cdot l}{\frac{1}{c^2}\cdot s^2-g^2-h^2-l^2} \tag{1.3.7}$$

The transfer-function of the pressure is Eq. (7).

2. Application

2.1. Solution of the one-dimension plane wave

The one-dimensional plane-wave can be described by deleting two transformed space-dimensions (e.g. y and z), thus their transformed version (in this case h and l) also disappear from Eq. (1.3.7):

$$P(s,g)=\frac{\left[\frac{1}{c^2}A_1+\rho_0 B_2\right]\cdot s-\left(A_2+\rho_0 B_1\right)\cdot g}{\frac{1}{c^2}\cdot s^2-g^2} \tag{2.1.1}$$

This transfer function has 2 poles. If we select the transformed time s so, that

$$s=j\omega$$

where $\omega=2\pi\cdot t$, the two poles are

$$g_{1,2}=\pm\frac{s}{c}=\pm\frac{j\omega}{c}\,. \tag{2.1.2}$$

With the partial fractions we can express the transformed pressure by

$$P(s,g)=\frac{Q(s,g_1)}{\left(g-g_1\right)}+\frac{Q(s,g_2)}{\left(g-g_2\right)} \tag{2.1.3}$$

After the inverse-transformation on g, for the pressure we get

$$P(\omega,x)=Q\left(s,g_1\right)\cdot e^{g_1 x}+Q\left(s,g_2\right)\cdot e^{g_2 x}$$

With Eq. (2) this becomes

$$P(\omega,x)=\frac{\frac{1}{c}A_1+\rho_0 c\cdot B_2\cdot j\omega-\left(A_2+\rho_0 B_1\right)\cdot j\omega}{-2j\omega}\cdot e^{\frac{j\omega}{c}x}+\frac{\frac{1}{c}A_1+\rho_0 c\cdot B_2\cdot j\omega+\left(A_2+\rho_0 B_1\right)\cdot j\omega}{2j\omega}\cdot e^{-\frac{j\omega}{c}x} \tag{2.1.4}$$

Since we are considering a plane-wave generated at $x=0$ and are supposing that the field doesn't have any effect on the radiation at $t=0$, the initial values are from Eqs. (1.3.3)

$$p(0,g) = A_1 = 0 \tag{2.1.5a}$$

$$p(s,0) = A_2 = P(s) \tag{2.1.5b}$$

$$v_x(0,g) = B_1 = 0 \tag{2.1.5c}$$

$$v_x(s,0) = B_2 = V_x(s) \tag{2.1.5d}$$

From these Eq. (4) simplifies to the form

$$P(\omega,x) = \frac{\rho_0 c \cdot V_x(\omega,0) - P(\omega,0)}{-2} \cdot e^{\frac{j\omega}{c}x} + \frac{\rho_0 c \cdot V_x(\omega,0) + P(\omega,0)}{2} \cdot e^{-\frac{j\omega}{c}x} \tag{2.1.6}$$

Writing Eq. (6) in another form

$$P(\omega,x) = \frac{1}{2}\left[P(\omega,0)\left(e^{-\frac{j\omega}{c}x} + e^{\frac{j\omega}{c}x} \right) + \rho_0 c \cdot V_x(\omega,0)\left(e^{-\frac{j\omega}{c}x} + e^{\frac{j\omega}{c}x} \right) \right] \tag{2.1.7}$$

If we denote the acoustical impedance of radiation at $x=0$ by

$$Z_r = \frac{P(\omega,0)}{V(\omega,0)} \qquad (2.1.7)$$

Eq. (6) gets the form

$$Z(\omega,x) = \rho_0 c \cdot \cos\left(\frac{\omega}{c}x\right) + j \cdot Z_r \sin\left(\frac{\omega}{c}x\right) \tag{2.1.8}$$

Or, without impedance

$$P(\omega,x) = \rho_0 c \cdot V_x(\omega,0)\cos\left(\frac{\omega}{c}x\right) + j \cdot P_x(\omega,0)\sin\left(\frac{\omega}{c}x\right) \tag{2.1.9}$$

To derive the time-domain solution, with the inverse Laplace-transformation we get

$$P(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\rho_0 c \cdot V_x(\omega,0)\cos\left(\frac{\omega}{c}x\right) + j \cdot P_x(\omega,0)\sin\left(\frac{\omega}{c}x\right) \right] \cdot e^{j\omega x} d\omega \tag{2.1.10}$$

Here V_x and P_x are arbitrary functions describing the excitation signals at $x=0$.

Conclusions

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References

[1] Allan D. Pierce - 'Acoustics - An Introduction to Its Physical Principles and Applications', 1991